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Application of the Fourier Method to The Study of Boundary Problems for Second and Third Order Elliptic-Hyperbolic Type Equations

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Abstract
Boundary problems for second and third-order elliptic-hyperbolic type equations are an important area of study in partial differential equations. These boundary value problems involve equations that exhibit both elliptic and hyperbolic behavior, leading to unique mathematical challenges.

Key words: *boundary problems, second-order, elliptic-hyperbolic type, hyperbolic behavior, equations, partial differential equations*

Boundary value problems for second and third-order elliptic-hyperbolic equations typically involve prescribing conditions on the boundaries of a domain where the equation is defined. These conditions can include Dirichlet boundary conditions, Neumann boundary conditions, or mixed boundary conditions. The goal is to find a solution that satisfies the equation within the domain while also satisfying the prescribed conditions on the boundary.

The study of such boundary problems requires a combination of techniques from elliptic and hyperbolic partial differential equations. Various mathematical tools, such as functional analysis, existence and uniqueness results, and specific solution techniques, are employed to analyze and solve these problems.

Specific examples of second and third-order elliptic-hyperbolic equations include the wave equation with mixed boundary conditions, the telegraph equation, and certain types of mixed type equations involving Laplacian and

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wave operators. These equations arise in diverse areas of mathematical physics, such as the study of wave propagation, signal transmission, and fluid dynamics. To explore this topic further and find references from Uzbekistan authors, I recommend searching academic databases, digital libraries, or the publications of mathematics departments at universities in Uzbekistan. Additionally, reaching out to mathematics professors or researchers in Uzbekistan who specialize in partial differential equations may provide valuable insights and references specific to the region.

Consider a mixed type equation

$$Lu \equiv u_{xx} + sign y u_{yy} - \rho^2 u(x,y) = 0 \qquad (1)$$

in a rectangular area $D=\{(x,y):\, 0<x<1,\, -p<y<q\}$, где $\rho,\ \ p>0,\ q>0$ – given real numbers,

Let us introduce the notation: $J=\{(x,y):0<x<1,\ y=0\}$,

$$D_1 = D \cap \{(x,y):\, x>0,\, y>0\}, \quad D_2 = D \cap \{(x,y):\, x>0,\, y<0\}, \quad D = D_1 \cup D_2 \cup J\ .$$

In the area we study the following problem.

Task 1. Find a function in the domain that satisfies the following conditions:

$$u(x,y) \in C^1(\overline{D}) \cap C^2(D_1 \cup D_2); \qquad (2)$$

$$Lu(x,y)=0, \quad (x,y)\in D_1 \cup D_2; \qquad (3)$$

$$u(0,y)=0, \qquad u(1,y)=0, \qquad -p \le y \le q, \qquad (4)$$

$$u(x,q)=\varphi(x), \quad u(x,-p)=\psi(x), \quad 0 \le x \le 1, \qquad (5)$$

where $\varphi(x), \quad \psi(x) -$ given sufficiently smooth functions, and

$$\varphi(0)=\varphi(1)=\psi(0)=\psi(1)=0. \qquad (6)$$

As we know, the Tricomi problem for equations of mixed parabolic-hyperbolic type has been studied by many authors in areas where the hyperbolic part is a triangle limited by the characteristics $x+y=0, \quad x-y=l, \quad l>0$ and type change line $y=0$.

In 1959 I.M. Gelfand proposed to study the problem of gas movement in a channel surrounded by a porous medium, while the gas movement in the channel was described by the wave equation, and outside it by the diffusion equation. In this work there was no mathematical formulation of the problem, and from the physical meaning of the proposed problem it follows that such a problem should be studied in a rectangular region, in connection with which K.B. Sabitov was the first to study the initial-boundary problem for the parabolic equation in hyperbolic type in a rectangular region.

Formal solution

We will look for partial solutions of equation (1) that are not equal to zero in the region D in the form

$$u(x,y)=X(x)Y(y), \qquad (7)$$

satisfying zero boundary conditions (4). Then calculating

$$X''(x)Y(y)+X(x)Y''(y)-\rho^2X(x)Y(y)=0,\ y>0 \quad \Big|: X(x)Y(y)\neq 0,$$

$$\frac{X''(x)}{X(x)}+\frac{Y''(x)}{Y(y)}-\rho^2=0, \quad \frac{X''(x)}{X(x)}=\rho^2-\frac{Y''(x)}{Y(y)}=-\mu^2;$$

$$X''(x)Y(y)-X(x)Y''(y)-\rho^2X(x)Y(y)=0,\ y<0 \quad \Big|: X(x)Y(y)\neq 0,$$

$$\frac{X''(x)}{X(x)}-\frac{Y''(x)}{Y(y)}-\rho^2=0, \quad \frac{X''(x)}{X(x)}=\rho^2+\frac{Y''(x)}{Y(y)}=-\mu^2$$

We will have

$$X''(x)+\mu^2X(x)=0, \quad 0\le x\le 1, \qquad (6)$$

$$X(0)=X(1)=0, \qquad (7)$$

$$Y''(y)-(\rho^2+\mu^2)Y(y)=0, \quad 0\le y\le q, \qquad (8)$$

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$$Y''(y)+(\rho^2+\mu^2)Y(y)=0, \quad -p\le y\le 0. \qquad (9)$$

Let's solve problems (6) and (7). The general solution to equation (6) has the form

$$X(x)=aCos\mu x+bSin\mu x,$$

$$\begin{aligned}&X(x)=aCos\mu x+bSin\mu x,\\&X(0)=aCos\mu 0+bSin\mu 0=a=0,\\&X(1)=aCos\mu+bSin\mu=bSin\mu=0, \quad b\ne 0, \quad Sin\mu=0\Rightarrow\\&\Rightarrow \mu=\pi n,\ n\in Z=\{-N,0,N\}.\end{aligned}$$

Then the spectral problem (6) and (7) has a solution

$$X_n(x)=\sqrt{2}\sin\pi nx, \quad \mu_n=\pi n, \quad n\in N. \qquad (10)$$

In this case, the system is orthonormal, complete and forms a basis in space $L_2[0,1]$[29, 31]. Then differential equations (8) and (9) have general solutions

$$Y_n(y)=\begin{cases}c_n e^{\lambda_n y}+d_n e^{-\lambda_n y}, & y>0,\\ a_n cos\lambda_n y+b_n sin\lambda_n y, & y<0,\end{cases} \qquad (11)$$

где $c_n,\ d_n,\ \ a_n,\ b_n-$ arbitrary constants, $\lambda_n=\sqrt{\mu_n^2+\rho^2}=\sqrt{(\pi n)^2+\rho^2}$.

For function (11), due to (2), the conjugation conditions are satisfied

$$Y_n(0+0)=Y_n(0-0), \quad Y_n'(0+0)=Y_n'(0-0). \qquad (12)$$

Now we find the coefficients $a_n,\ b_n,\ c_n$ и d_n. By virtue of (12), from (11) we obtain

$$\begin{cases}c_n+d_n=a_n,\\ c_n+d_n=b_n,\end{cases}\Rightarrow c_n=\frac{a_n+b_n}{2}, \quad d_n=\frac{a_n-b_n}{2}. \qquad (13)$$

Substituting (13) into (11) we have

$$Y_n(y)=\begin{cases}\dfrac{a_n+b_n}{2}e^{\lambda_n y}+\dfrac{a_n-b_n}{2}e^{-\lambda_n y}, & y>0,\\ a_n cos\lambda_n y+b_n sin\lambda_n y, & y<0.\end{cases} \quad (14)$$

Using series expansions in terms of sines of the function

$$\psi(x)=\sum_{n=1}^{+\infty}\psi_n X_n(x),\quad \varphi(x)=\sum_{n=1}^{+\infty}\varphi_n X_n(x) \quad (15)$$

taking into account (5) from (14) with $0<y<q$ и $-p<y<0$ we have

$$\begin{cases}Y_n(q)=\dfrac{a_n+b_n}{2}e^{\lambda_n q}+\dfrac{a_n-b_n}{2}e^{-\lambda_n q}=\varphi_n, & y>0,\\ Y_n(-p)=a_n cos\lambda_n p-b_n sin\lambda_n p=\psi_n, & y<0,\end{cases}$$

$$\begin{cases}a_n ch\lambda_n q+b_n sh\lambda_n q=\varphi_n, & |\times Cos\lambda_n p\\ a_n cos\lambda_n p-b_n sin\lambda_n p=\psi_n, & |\times ch\lambda_n q\end{cases} -$$

$$\begin{cases}a_n ch\lambda_n q+b_n sh\lambda_n q=\varphi_n, & |\times sin\lambda_n p\\ a_n cos\lambda_n p-b_n sin\lambda_n p=\psi_n, & |\times sh\lambda_n q\end{cases} + \Rightarrow$$

$$a_n=\frac{\varphi_n sin\lambda_n p+\psi_n sh\lambda_n q}{\Delta_{pq}(n)},\quad b_n=\frac{\varphi_n cos\lambda_n p-\psi_n ch\lambda_n q}{\Delta_{pq}(n)},\quad n\in N, \quad (16)$$

Substituting (16) into (14) we find

$$Y_n(y)=\frac{a_n+b_n}{2}e^{\lambda_n y}+\frac{a_n-b_n}{2}e^{-\lambda_n y}=a_n ch\lambda_n y+b_n sh\lambda_n y=$$

$$=\frac{1}{\Delta_{pq}(n)}\left[\varphi_n sin\lambda_n p ch\lambda_n y+\psi_n sh\lambda_n q ch\lambda_n y+\varphi_n cos\lambda_n p sh\lambda_n y-\psi_n ch\lambda_n q sh\lambda_n y\right]=$$

$$=\frac{1}{\Delta_{pq}(n)}\left[\psi_n sh\lambda_n (q-y)+\varphi_n sin\lambda_n p ch\lambda_n y+\varphi_n cos\lambda_n p sh\lambda_n y\right],\ y>0, \quad (18)$$

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$$Y_n(y)=a_n cos\lambda_n y+b_n sin\lambda_n y=\frac{1}{\Delta_{pq}(n)}[\varphi_n sin\lambda_n pcos\lambda_n y+$$

$$+\psi_n sh\lambda_n qcos\lambda_n y+\varphi_n cos\lambda_n psin\lambda_n y-\psi_n ch\lambda_n qsin\lambda_n y]=$$

$$=\frac{1}{\Delta_{pq}(n)}[\varphi_n sin\lambda_n(p+y)+\psi_n sh\lambda_n qcos\lambda_n y-\psi_n ch\lambda_n qsin\lambda_n y],\ y<0. \quad (19)$$

Due to the linearity and homogeneity of equation (1), taking into account (7), (10), (18), (19), we find a formal solution to problem 1 in the form

$$u(x,y)=\sqrt{2}\sum_{n=1}^{+\infty}\frac{1}{\Delta_{pq}(n)}\Big[\psi_n sh\lambda_n(q-y)+\varphi_n sin\lambda_n pch\lambda_n y+$$

$$+\varphi_n cos\lambda_n psh\lambda_n y\Big]sin\pi nx,\quad 0<y<q, \quad (20)$$

$$u(x,y)=\sqrt{2}\sum_{n=1}^{+\infty}\frac{1}{\Delta_{pq}(n)}[\varphi_n sin\lambda_n(p+y)+\psi_n sh\lambda_n qcos\lambda_n y-$$

$$-\psi_n ch\lambda_n qsin\lambda_n y]sin\pi nx,\quad -p<y<0, \quad (21)$$

Uniqueness of solution to the problem

Theorem 1. If there is a solution to Problem 1, then it is unique only if condition (17) is satisfied for all $n\in N$.

Proof of Theorem 1. *Let there be two solutions* $u_1(x,y)$ и $u_2(x,y)$ problem 1. Then their difference $u(x,y)=u_1(x,y)-u_2(x,y)$ satisfies equation (1) and homogeneous conditions:

$$u(x,-0)=u(x,+0),\quad (x,0)\in \bar{J}, \quad (22)$$

$$\lim_{y\to+0}u_y(x,y)=\lim_{y\to-0}u_y(x,y),\quad (x,0)\in J, \quad (23)$$

$$u(0,y)=0,\qquad u(1,y)=0,\qquad -p\le y\le q, \quad (24)$$

$$u(x,-p)=0, \quad u(x,q)=0, \quad 0\le x\le 1 . \tag{25}$$

It is known that function (10) forms a complete orthonormal system in *L2* [29,31]. Following [30], [41], we consider the integral

$$u_n(y)=\int_0^1 u(x,y)X_n(x)dx, \quad -p\le y\le q. \tag{26}$$

Based on (26), we introduce the function

$$u_{n,\varepsilon}(y)=\int_\varepsilon^{1-\varepsilon} u(x,y)X_n(x)dx, \tag{27}$$

where $\varepsilon -$ quite a small number.

Differentiating equality (27) with respect to $y\in(-p,0)\cup(0,q)$ twice and taking into account equation (1), we get

$$u''_{n,\varepsilon}(y)=\int_\varepsilon^{1-\varepsilon} u_{yy}(x,y)X_n(x)dx=$$

$$=\int_\varepsilon^{1-\varepsilon}\left[\rho^2u(x,y)-u_{xx}(x,y)\right]X_n(x)dx, \quad 0<y<q, \tag{28}$$

$$u''_{n,\varepsilon}(y)=\int_\varepsilon^{1-\varepsilon} u_{yy}(x,y)X_n(x)dx=\int_\varepsilon^{1-\varepsilon}\left[u_{xx}(x,y)-\rho^2u(x,y)\right]X_n(x)dx, \quad -p<y<0. \tag{29}$$

From here the right-hand side of equality (28) and (29), integrating by parts twice and passing to the limit at taking into account (24), we obtain

$$u''_n(y)-\lambda_n^2u_n(y)=0, \quad 0<y<q, \tag{30}$$

$$u''_n(y)+\lambda_n^2u_n(y)=0, \quad -p<y<0, \tag{31}$$

где $\lambda_n^2=\rho^2+\mu_n^2=\rho^2+\pi^2n^2.$

Then differential equations (30) and (31) have general solutions

$$u_n(y)=\begin{cases} a_n ch\lambda_n y + b_n sh\lambda_n y, & y>0,\\ a_n cos\lambda_n y + b_n sin\lambda_n y, & y<0, \end{cases} \quad (32)$$

where $a_n,\ b_n -$ is identified by (16).

So, the functions $u_n(y)$ are uniquely defined, which allows us to prove Theorem 1 of Problem 1. Let $u(x,y)-$ solution of the homogeneous problem (22)-(25) and conditions (17) are satisfied for all $n\in N$. Then, $\varphi_n=\psi_n\equiv 0$ and from formulas (32) and (26) taking into account (16) it follows that for any $y\in[-p,q]$

$$\int_0^1 u(x,y)X_n(x)dx=0, \quad -p\le y\le q, \quad n\in N=\{1,2,.....\}. \quad (33)$$

Hence, due to the completeness of the system $\left\{\sqrt{2}\sin\pi nx\right\}_{n=1}^{\infty}$ in space $L_2[0,1]$ [56] follows that $u(x,y)=0$ almost everywhere on $[0,1]$ at any $y\in[-p,q]$. Since, by virtue of (2), the function $u(x,y)$ continuous in $\bar{D}$, then $u(x,y)\equiv 0$ в $\bar{D}$.
Therefore, the solution to Problem 1 is unique if the condition is satisfied (17).
Theorem 1 is proven.
Inverse problems arise in a variety of areas of human activity, such as physics (inverse problems of quantum scattering theory), geophysics (inverse problems of electrical prospecting, seismic, potential theory), biology, medicine, quality control of industrial products, etc. More detailed information related to inverse problems for partial differential equations can be found in.
Inverse problems for an equation of mixed type were studied in the works of K.B. Sabitov, here and unknown functions. Following these works, various inverse problems for a series of equations of mixed type of an integer order were studied in the works of A.V. Yuldasheva, S. Dzhamalov.
Inverse problems for equations of mixed type with a fractional order operator are poorly studied. In this direction, we note the works of B.Zh. Kadirkulov, E.T. Karimov. In for equation (72), inverse problems with periodicity conditions in a rectangular domain were studied.

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